

# Black Hole Decay and Quantum Instantons

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## Abstract

We study the analytic structure of the S-matrix which is obtained from the reduced Wheeler-DeWitt wave function describing spherically symmetric gravitational collapse of massless scalar fields. The complex simple poles in the S-matrix lead to the wave functions that satisfy the same boundary condition as quasi-normal modes of a black hole, and correspond to the bounded states of the Euclidean Wheeler-DeWitt equation. These wave function are interpreted as quantum instantons.

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In the previous work [1] we studied quantum mechanically the self-similar black hole formation by collapsing scalar fields and found the wave functions that give the correct semi-classical limit. The reduced Wheeler-DeWitt equation for gravity belongs to an exactly solvable Calogero type system with an inverted potential whose attractive inverse square and repulsive square potential terms give rise to a potential barrier. The boundary condition for black hole formation was that the wave function has both the incoming and the outgoing flux at spatial infinity and only the incoming flux toward the black hole singularity.

Of particular interest is the subcritical case, in which a black hole can be formed through quantum tunneling. Due to the time reversal symmetry, however, the subcritical wave function may be given an interpretation of the reversal process of black hole formation, that is, the decay of the black hole [2]. Then the wave function for black hole decay should have a purely outgoing flux. This wave function is somehow reminiscent of the gravitational wave from a perturbation of a black hole [3]. Moreover, for a certain discrete spectrum of complex frequencies there occur quasi-normal modes that have both purely outgoing modes at spatial infinity and purely incoming ones at the horizon of the black hole [4].

In this paper we study the pole structure of the S-matrix which is obtained from the wave function for black hole formation. The boundary condition that the wave function should have a purely outgoing flux at spatial infinity and a purely incoming one at the classical apparent horizon leads to a discrete spectrum of complex parameters ( $c_0$ ). It is further shown that this wave function can be obtained through the analytical continuation of a bounded state of the corresponding Euclidean Wheeler-DeWitt equation. Just as quasi-normal modes of perturbations of a black hole can be interpreted as instantons [5], these exact wave functions of the quantum theory for black hole decay may be interpreted as quantum instantons

The spherically symmetric geometry minimally coupled to a massless scalar field is described by the reduced action in  $(1+1)$ -dimensional spacetime of which the Hilbert-Einstein action is

$$S = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left[ R - 2(\nabla\phi)^2 \right] + \frac{1}{8\pi} \int_{\partial M} d^3x K \sqrt{h}. \quad (1)$$

The reduced action is

$$S_{sph} = \frac{1}{4} \int d^2x \sqrt{-\gamma} r^2 \left[ \left\{ {}^{(2)}R(\gamma) + \frac{2}{r^2} \left( (\nabla r)^2 + 1 \right) \right\} - 2(\nabla\phi)^2 \right], \quad (2)$$

where  $\gamma_{ab}$  is the  $(1+1)$ -dimensional metric. The spherical spacetime metric is

$$ds^2 = -2du dv + r^2 d\Omega_2^2, \quad (3)$$

where  $d\Omega_2^2$  is the usual spherical part of the metric, and  $u$  and  $v$  are null coordinates. The self-similarity condition is imposed such that

$$r = \sqrt{-uv} y(z), \quad \phi = \phi(z), \quad (4)$$

where  $z = +v/(-u) = e^{-2\tau}$ ,  $y$  and  $\phi$  depend only on  $z$ . We introduce another coordinates  $(\omega, \tau)$

$$u = -\omega e^{-\tau}, \quad v = \omega e^{\tau}, \quad (5)$$

to rewrite the metric as

$$ds^2 = -2N^2(\tau)\omega^2 d\tau^2 + 2d\omega^2 + \omega^2 y^2 d\Omega_2^2, \quad (6)$$

where  $N(\tau)$  is the lapse function of the ADM formulation.

The classical solutions of the field equations were obtained by Roberts [6], and studied in connection with gravitational collapse by others [7]. Classically black hole formation is only allowed in the supercritical case ( $c_0 > 1$ ), but even in the subcritical situation there are quantum mechanical tunneling processes to form a black hole of which the probability is semiclassically calculated [8,1].

In our previous work [1] we quantized the system canonically with the ADM formulation to obtain the Wheeler-DeWitt equation for the quantum black hole formation

$$\left[ \frac{\hbar^2}{2K} \frac{\partial^2}{\partial y^2} - \frac{\hbar m_P^2}{2Ky^2} \frac{\partial^2}{\partial \phi^2} - K \left( 1 - \frac{y^2}{2} \right) \right] \Psi(y, \phi) = 0, \quad (7)$$

where  $K/\hbar \equiv (m_P^2/\hbar^2)(\omega_c^2/2)$  plays the role of a cut-off parameter of the model, and we use a unit system  $c = 1$ . The wave function can be factorized to the scalar and gravitational parts,

$$\Psi(y, \phi) = \exp \left( \pm i \frac{Kc_0}{\hbar^{1/2} m_P} \phi \right) \psi(y). \quad (8)$$

Here the scalar field part is chosen to yield the classical momentum  $\pi_\phi = \hbar K y^2 \dot{\phi} / m_P^2 N = \pm Kc_0$ , where  $c_0$  is the dimensionless parameter determining the supercritical ( $c_0 > 1$ ), the critical ( $c_0 = 1$ ), and the subcritical ( $1 > c_0 > 0$ ) collapse.

Now the gravitational field equation of the Wheeler-DeWitt equation takes the form of a Schrödinger equation

$$\left[ \frac{-\hbar^2}{2K} \frac{d^2}{dy^2} + \frac{K}{2} \left( 2 - y^2 - \frac{c_0^2}{y^2} \right) \right] \psi(y) = 0. \quad (9)$$

The solution describing black hole formation was obtained in Ref. [1]:

$$\psi_{BH}(y) = \left[ \exp \left( \frac{-iK}{2\hbar} y^2 \right) \right] \left( \frac{K}{\hbar} y^2 \right)^\mu M \left( a, b, i \frac{K}{\hbar} y^2 \right), \quad (10)$$

where  $M$  is the confluent hypergeometric function and

$$a = \frac{1}{2} - \frac{i}{2\hbar}(Q + K), \quad b = 1 - \frac{i}{\hbar}Q, \quad \mu = \frac{1}{4} - \frac{i}{2\hbar}Q \quad (11)$$

with

$$Q = \left( K^2 c_0^2 - \frac{\hbar^2}{4} \right)^{1/2}. \quad (12)$$

Using the asymptotic form [9] at spatial infinity

$$\psi_{BH}(y) \simeq \frac{\Gamma(b)}{\Gamma(b-a)} e^{i\pi a} \left(i \frac{K}{\hbar} y^2\right)^{\mu-a} e^{-(i/2)(K/\hbar)y^2} + \frac{\Gamma(b)}{\Gamma(a)} \left(i \frac{K}{\hbar} y^2\right)^{\mu+a-b} e^{(i/\hbar)(K/\hbar)y^2}, \quad (13)$$

we obtain the S-matrix component describing the reflection rate

$$S = \frac{\Gamma(b-a)}{\Gamma(a)} \frac{(iK/\hbar)^{2a-b}}{e^{i\pi a}}. \quad (14)$$

From the S-matrix follows the transmission rate for black hole formation

$$\begin{aligned} \frac{j_{trans}}{j_{in}} &= 1 - |S|^2 \\ &= 1 - \frac{\cosh \frac{\pi}{2\hbar}(Q+K)}{\cosh \frac{\pi}{2\hbar}(Q-K)} e^{-(\pi/\hbar)Q}, \end{aligned} \quad (15)$$

where  $\left|\Gamma\left(\frac{1}{2} + ix\right)\right|^2 = \frac{\pi}{\cosh(\pi x)}$  is used. Equation (15) gives the probability of black hole formation for the supercritical, critical, and subcritical  $c_0$ -values.

We now consider the analytic structure of the S-matrix: it is an analytic function of  $Q$  and  $K$  with simple poles which can be explicitly shown as

$$S = \sum_{N=0}^{\infty} \frac{1}{(Q-K)/\hbar + i(2N+1)} \left( \frac{2ie^{-(\pi/2\hbar)K - i(K/\hbar)\ln(K/\hbar)}}{N!\Gamma(-N - i(K/\hbar))} \right). \quad (16)$$

The poles reside in the unphysical region of the parameter space of  $Q$  and  $K$ :

$$Q - K = -i\hbar(2N+1), \quad (N = 0, 1, 2, \dots). \quad (17)$$

It should be remarked that these poles make the first term of Eq. (13) vanish since

$$b - a = -\frac{i}{2\hbar}(Q - K) + \frac{1}{2} = -N, \quad (N = 0, 1, 2, \dots). \quad (18)$$

The second term of Eq. (13) has a purely outgoing flux at spatial infinity. The wave function near the apparent horizon, which can be obtained by the steepest descent method in the Appendix of Ref. [1] and by taking the large  $(K/\hbar)$ -limit, leads to the flux

$$j_{AH} \simeq A^2(y) \left\{ \frac{1}{2} y(1-y^2) \left[ \frac{(y^4 + c^{*2} - 2y^2)^{1/2} + (y^4 + c^2 - 2y^2)^{1/2}}{\left((y^4 + c^{*2} - 2y^2)(y^4 + c^2 - 2y^2)\right)^{1/2}} \right] - \frac{1}{2}(c_0^* + c_0) \frac{1}{y} \right\}, \quad (19)$$

where  $A(y)$  denotes an amplitude, a real function, and  $c_0 = 1 - i(\hbar/K)(2N+1)$  from Eq. (17) and  $c = (c_0^2 - \hbar^2/4K^2)^{1/2} - i(\hbar/K)$ . At the apparent horizon  $y_{AH} = c_0/\sqrt{2}$ , the wave function has an incoming flux. Therefore, the poles are the outcome of the same boundary condition used to find quasi-normal modes of a black hole [4]. Note that the wave function (10) has also the purely incoming flux toward the black hole singularity at  $y = 0$ .

A few comments are in order. First, for physical processes of gravitational collapse there can not be poles because  $K$  and  $c_0$  are real-valued. In ordinary quantum mechanics, the poles of S-matrix occur at the bound states [10], and in relativistic scattering at the resonances or the Regge poles [11]. Our case is analogous to a meta-stable quantum mechanical system of

which poles are identified with quasi-stationary states that describe the decay of a particle through a potential barrier. Second, we calculated quantum decay rate of a black hole as a reversed process of gravitational collapse through a barrier by quantum tunneling. This quantum decay process, first studied by Tomimatsu [2], is a distinctively different decay channel from the Hawking radiation process. It will be interesting to investigate both processes present in one model. Third, it should be pointed out that our discussion based upon the similarity of the boundary condition on the wave function with quasi-normal modes seems to have no deeper physical connection more than analogy because our model works only for a dynamical stage of gravitational collapse and its reversed process, rather than the quasi-stationary stage at late times.

Recalling that the poles in Eqs. (16) or (18) result from the potential barrier and that the exponential behavior of the Wheeler-DeWitt equation under a potential barrier describes a Euclidean geometry, we turn to the Euclidean theory of gravitational collapse. In the Euclidean theory the Wheeler-DeWitt equation has oscillatory wave functions and a well-defined semiclassical limit even under the potential barrier of the Lorentzian theory [12]. The Euclidean geometry with the metric

$$ds_E^2 = 2N^2(\tau)\omega^2 d\tau_E^2 + 2d\omega^2 + \omega^2 y^2 d\Omega_2^2, \quad (20)$$

leads to the Wheeler-DeWitt equation

$$\left[ -\frac{\hbar^2}{2K} \frac{\partial^2}{\partial y^2} + \frac{\hbar m_P^2}{2Ky^2} \frac{\partial^2}{\partial \phi^2} - K \left( 1 - \frac{y^2}{2} \right) \right] \Psi_E(y, \phi) = 0. \quad (21)$$

According to the transformation rule  $i\pi_\phi \leftrightarrow \pi_{E,\phi}$  of the scalar field momenta between the Lorentzian and Euclidean geometries [12], the wave function has the form

$$\Psi_E(y, \phi) = \exp \left( \mp \frac{Kc_0}{\hbar^{1/2} m_P} \phi \right) \psi_E(y). \quad (22)$$

The Wheeler-DeWitt equation reduces to the gravitational field equation

$$\left[ \frac{-\hbar^2}{2K} \frac{d^2}{dy^2} + \frac{K}{2} \left( y^2 + \frac{c_0^2}{y^2} - 2 \right) \right] \psi_E(y) = 0. \quad (23)$$

Notice that this is a variant of Calogero models with the Calogero-Moser Hamiltonian [13], but the energy eigenvalue is fixed, and only a quantized  $c_0$  is allowed. Since Eq. (23) can also be obtained from the Lorentzian equation (9) by letting

$$K = iK_E, \quad (24)$$

one may obtain, through the analytical continuation of Eq. (10), the solution to Eq. (23)

$$\psi_E(y) = \left[ \exp \left( \frac{1}{2} \frac{K_E}{\hbar} y^2 \right) \right] \left( \frac{K_E}{\hbar} y^2 \right)^{\mu_E} M \left( a_E, b_E, -\frac{K_E}{\hbar} y^2 \right), \quad (25)$$

where

$$a_E = \frac{1}{2} + \frac{1}{2\hbar}(Q_E + K_E), \quad b_E = 1 + \frac{Q_E}{\hbar}, \quad \mu_E = \frac{1}{4} + \frac{1}{2\hbar}Q_E \quad (26)$$

with

$$Q_E = \left( K_E^2 c_0^2 + \frac{\hbar^2}{4} \right)^{1/2}. \quad (27)$$

The asymptotic form of Eq. (25) leads to the bounded states only when

$$b_E - a_E = -N, \quad (N = 0, 1, 2, \dots), \quad (28)$$

that is, the condition is satisfied

$$Q_E - K_E = -\hbar(2N + 1). \quad (29)$$

The condition (29) is identical to the pole position of the S-matrix with  $K = iK_E$  given in Eqs. (17) and (18).

A few remarks are in order. First, the quantum solution (25) is analogous to an instanton in the sense it is a solution in the Euclidean sector, but is not in the strict sense because the Wheeler-DeWitt equation is already a quantum equation, not a classical one. The semiclassical result from the Bohr-Sommerfeld quantization rule

$$\frac{\pi}{2} \frac{K_E}{\hbar} (1 - c_0) = \pi \left( N + \frac{1}{2} \right), \quad (N = 0, 1, 2, \dots), \quad (30)$$

is the large  $(K_E/\hbar)$ -limit of the exact result (29). The instantons, the left hand side of Eq. (30), provides semiclassically the probability of tunneling process [8]. Second, the correspondence between the poles and the Euclidean polynomial solutions breaks down for large  $N$ . While the poles contribute for all  $N$  without limit, the normalizable Euclidean solutions exist only for  $N < K_E/2\hbar$ . The polynomial solutions for large  $N$  are well defined, but are not normalizable. We have not yet understood these nonnormalizable solutions. Finally, we consider the classical field equations corresponding to the poles of the S-matrix. In the Lorentzian geometry the relevant equations are

$$\frac{d\phi}{d\tau} = \frac{c_0}{y^2}, \quad (31)$$

$$\left( \frac{dy}{d\tau} \right)^2 = K^2 \left( -2 + y^2 + \frac{c_0^2}{y^2} \right), \quad (32)$$

where  $c_0 \simeq 1 - i\hbar(2N + 1)/K$ , for large  $K$ . The complex  $c_0$  implies complex  $\frac{d\phi}{d\tau}$  and  $\frac{dy}{d\tau}$ , which may be imagined as a bound state like complex momentum in quantum mechanics and requires a complex spacetime metric. In the Euclidean geometry ( $K = iK_E$ ) these classical equations are the same as those equations with quantized  $c_0$  in the tunneling region in Ref. [8].

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